DÚ č. 5 - riešenie

1. Vyšetrite priebeh funkcie $f(x) = \frac{\cos x}{\cos 2x}$ a nakreslite jej graf.

   $D(f) = \mathbb{R} \setminus \{\frac{\pi}{4} + \frac{k\pi}{2}, k \in \mathbb{Z}\}$

   Funkcia je periodická s periódou 2π, preto stačí jej priebeh vyšetriť na intervale $\left(\frac{\pi}{4}, \frac{9\pi}{4}\right)$.

   $P_y = [0; 1]$

   $P_{x_1} = [\frac{\pi}{2}; 0]$

   $P_{x_2} = [\frac{3\pi}{2}; 0]$

   $\lim\limits_{x \to \frac{\pi}{4}^+} \frac{\cos x}{\cos 2x} = -\infty$

   $\lim\limits_{x \to \frac{3\pi}{4}^-} \frac{\cos x}{\cos 2x} = \infty$

   $\lim\limits_{x \to \frac{3\pi}{4}^+} \frac{\cos x}{\cos 2x} = -\infty$

   $\lim\limits_{x \to \frac{5\pi}{4}^-} \frac{\cos x}{\cos 2x} = -\infty$

   $\lim\limits_{x \to \frac{5\pi}{4}^+} \frac{\cos x}{\cos 2x} = \infty$

   $\lim\limits_{x \to \frac{5\pi}{4}^-} \frac{\cos x}{\cos 2x} = -\infty$

   $\lim\limits_{x \to \frac{5\pi}{4}^+} \frac{\cos x}{\cos 2x} = \infty$

   $\lim\limits_{x \to \frac{5\pi}{4}^-} \frac{\cos x}{\cos 2x} = \infty$

   $f'(x) = \frac{\sin x\left(1+2\cos^2 x\right)}{\cos^2 2x}$

	$x \in \left(\frac{\pi}{4}; \frac{3\pi}{4}\right)$	$x \in \left(\frac{3\pi}{4}; \pi\right)$	$x \in \left(\pi; \frac{5\pi}{4}\right)$	$x \in \left(\frac{5\pi}{4}; \frac{7\pi}{4}\right)$	$x \in \left(\frac{7\pi}{4}; 2\pi\right)$	$x \in \left(2\pi; \frac{9\pi}{4}\right)$
$f'(x)$	$+$	$+$	$-$	$-$	$-$	$+$

   lokálne maximum: $[\pi; -1]$

   lokálne minimum: $[2\pi; 1]$

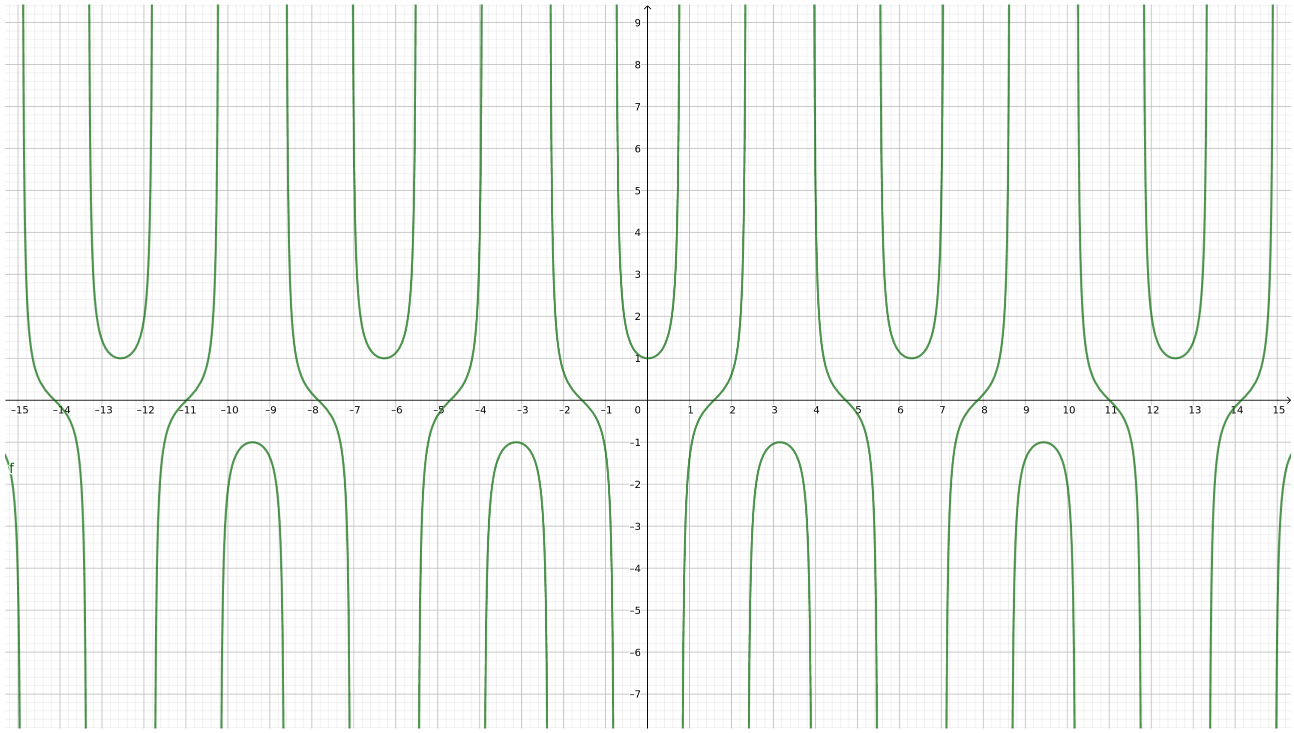
   $f'(\frac{\pi}{2}) = -1 \;\Rightarrow$ dotyčnica ku grafu funkcie v bode $[\frac{\pi}{2}, 0]$ zviera s kladným smerom osi x uhol $135°$

   $f'(\frac{3\pi}{2}) = -1 \;\Rightarrow$ dotyčnica ku grafu funkcie v bode $[\frac{3\pi}{2}, 0]$ zviera s kladným smerom osi x uhol $135°$

   $f''(x) = \frac{\cos x\left(4\sin^4 x + 7\sin^2 x + \cos^2 x + 12\sin^2 x\cos^2 x\right)}{\cos^3(2x)}$

	$x \in \left(\frac{\pi}{4}; \frac{\pi}{2}\right)$	$x \in \left(\frac{\pi}{2}; \frac{3\pi}{4}\right)$	$x \in \left(\frac{3\pi}{4}; \frac{5\pi}{4}\right)$	$x \in \left(\frac{5\pi}{4}; \frac{3\pi}{2}\right)$	$x \in \left(\frac{3\pi}{2}; \frac{7\pi}{4}\right)$	$x \in \left(\frac{7\pi}{4}; \frac{9\pi}{4}\right)$
$f''(x)$	$-$	$+$	$-$	$+$	$-$	$+$

   inflexné body: $\left[\frac{\pi}{2}; 0\right]$, $\left[\frac{3\pi}{2}; 0\right]$



Obrázek 1: $f(x) = \frac{\cos x}{\cos 2x}$

2. Vyšetrite priebeh funkcie $f(x) = (x - 2\,\mathrm{arctg}\,(x-5)) \cdot \mathrm{sgn}\, x$ a nakreslite jej graf.

$$f(x) = \begin{cases} 2\,\mathrm{arctg}\,(x-5) - x & \text{pre } x < 0, \\ 0 & \text{pre } x = 0, \\ x - 2\,\mathrm{arctg}\,(x-5) & \text{pre } x > 0. \end{cases}$$

$D(f) = \mathbb{R}$

$P_x = [a; 0]$, kde $a \approx -2{,}9$

$P_y = [0; 0]$

$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} (2\,\mathrm{arctg}\,(x-5) - x) = 2\,\mathrm{arctg}(-5) \approx -2{,}7$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} x - 2\,\mathrm{arctg}\,(x-5) = -2\,\mathrm{arctg}(-5) \approx 2{,}7$

$k = \lim\limits_{x\to\infty} \frac{f(x)}{x} = \lim\limits_{x\to\infty} \left(1 - \frac{2\,\mathrm{arctg}(x-5)}{x}\right) = 1$

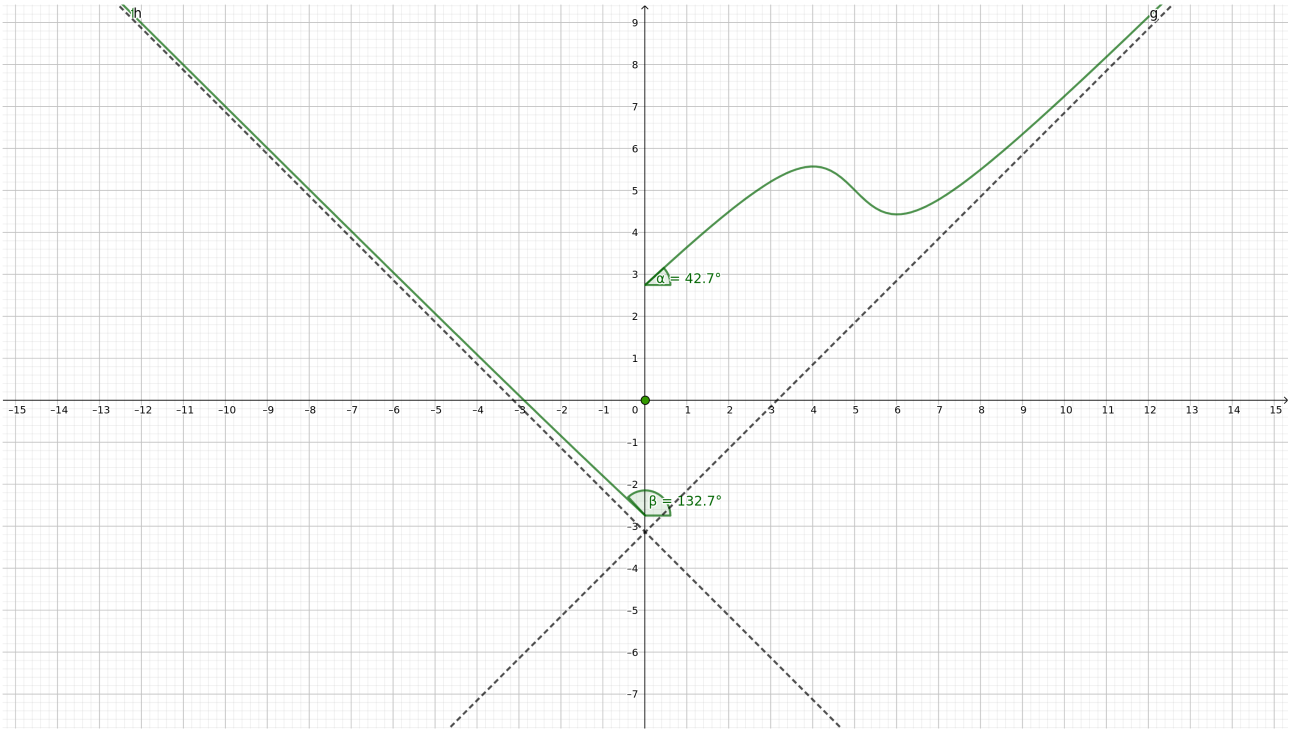
$q = \lim\limits_{x\to\infty} (f(x) - kx) = \lim\limits_{x\to\infty} (-2\,\mathrm{arctg}(x-5)) = -\pi$

$k' = \lim\limits_{x\to-\infty} \frac{f(x)}{x} = \lim\limits_{x\to-\infty} \left(\frac{2\,\mathrm{arctg}(x-5)}{x} - 1\right) = -1$

$q' = \lim\limits_{x\to-\infty} (f(x) - k'x) = \lim\limits_{x\to-\infty} (2\,\mathrm{arctg}(x-5)) = -\pi$

asymptoty: $y = x - \pi$, $y = -x - \pi$

$$f'(x) = \begin{cases} -\frac{x^2-10x+24}{x^2-10x+26} & \text{pre } x < 0, \\ \frac{x^2-10x+24}{x^2-10x+26} & \text{pre } x > 0. \end{cases}$$



Obrázek 2: $f(x) = (x - 2\,\mathrm{arctg}\,(x-5)) \cdot \mathrm{sgn}\, x$

$\lim\limits_{x\to 0^+} \frac{x^2-10x+24}{x^2-10x+26} = \frac{12}{13} \;\Rightarrow \alpha = \mathrm{arctg}\left(\frac{12}{13}\right) \approx 42{,}7^\circ$ (viď graf)

$\lim\limits_{x\to 0^-} -\frac{x^2-10x+24}{x^2-10x+26} = -\frac{12}{13} \;\Rightarrow \beta \approx 132{,}7^\circ$

	$x \in (-\infty; 0)$	$x \in (0; 4)$	$x \in (4; 6)$	$x \in (6; \infty)$
$f'(x)$	$-$	$+$	$-$	$+$

lokálne maximum: $[4; 4 + \frac{\pi}{2}]$

lokálne minimum: $[6; 6 - \frac{\pi}{2}]$

$$f''(x) = \begin{cases} -\frac{4x-20}{(x^2-10x+26)^2} & \text{pre } x < 0, \\ \frac{4x-20}{(x^2-10x+26)^2} & \text{pre } x > 0. \end{cases}$$

	$x \in (-\infty; 0)$	$x \in (0; 5)$	$x \in (5; \infty)$
$f''(x)$	$+$	$-$	$+$

inflexný bod: $[5; 5]$

3. Pomocou Taylorovho polynómu vypočítajte približnú hodnotu $\sqrt[3]{30}$ s chybou menšou ako 10^{-5}.

   Zostrojíme Taylorov polynóm funkcie $f(x) = \sqrt[3]{x}$ v bode $x_0 = 27$. Potom $\sqrt[3]{30} = f(30) \approx T_3(30)$. Platí, že $f^{(n+1)}(x) = (-1)^n \cdot \frac{2\cdot 5\cdot\ldots\cdot(3n-1)}{3^{n+1}} x^{-\frac{3n+2}{3}}$. Preto

$$|R_{n+1}(30)| = \frac{2\cdot 5\cdot \ldots \cdot (3n-1)\cdot 3^{n+1}}{3^{n+1}\xi^{\frac{3n+2}{3}}(n+1)!} = \frac{2\cdot 5\cdot \ldots \cdot (3n-1)}{\xi^{\frac{3n+2}{3}}(n+1)!},$$

kde $27 < \xi < 30$. Platí

$$|R_{n+1}(30)| = \frac{2 \cdot 5 \cdot \ldots \cdot (3n-1)}{\xi^{\frac{3n+2}{3}}(n+1)!} < \frac{2 \cdot 5 \cdot \ldots \cdot (3n-1)}{27^{\frac{3n+2}{3}}(n+1)!} < \frac{2 \cdot 5 \cdot \ldots \cdot (3n-1)}{3^{3n+2}(n+1)!}.$$

Hľadáme také n, že $\frac{2 \cdot 5 \cdot \ldots \cdot (3n-1)}{3^{3n+2}(n+1)!} < 10^{-5}$. To platí pre $n = 4$, preto musíme zostrojiť Taylorov polynóm štvrtého stupňa:

$$T_4(x) = 3 + \frac{1}{27}(x-27) - \frac{1}{2187}(x-27)^2 + \frac{5}{531441}(x-27)^3 - \frac{10}{43046721}(x-27)^4.$$

Potom platí

$$\sqrt[3]{30} \approx T_4(30) = 3 + \frac{1}{9} - \frac{1}{243} + \frac{5}{19683} - \frac{10}{531441}.$$

4. Pomocou Taylorovho polynómu vypočítajte približnú hodnotu e s chybou menšou ako 10^{-5}.

   Zostrojíme Taylorov polynóm funkcie $f(x) = e^x$ v bode $x_0 = 0$. Potom $e = f(1) \approx T(1)$. Platí, že $f^{(n+1)}(x) = e^x$. Preto $R_{n+1}(1) = \frac{e^\xi}{(n+1)!}$, kde $0 < \xi < 1$. Platí

$$|R_{n+1}(1)| = \frac{e^\xi}{(n+1)!} < \frac{e}{(n+1)!} < \frac{3}{(n+1)!}.$$

   Hľadáme také n, že $\frac{3}{(n+1)!} < 10^{-5}$. To platí pre $n = 8$, preto zostrojíme Taylorov polynóm ôsmeho stupňa:

$$T_8(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \frac{x^8}{40320}.$$

   Potom platí

$$e \approx T_8(1) = 2 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} + \frac{1}{40320}.$$

5.

$$\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{e^x - 1}\right) = \lim_{x\to 0}\frac{e^x - 1 - \sin x}{(e^x-1)\sin x} \overset{\text{L'H}}{=} \lim_{x\to 0}\frac{e^x - \cos x}{(e^x-1)\cos x + e^x \sin x} \overset{\text{L'H}}{=}$$
$$= \lim_{x\to 0}\frac{e^x + \sin x}{\sin x + 2e^x \cos x} = \frac{1}{2}$$

6.

$$\lim_{x\to 0}(\cos x)^{\frac{1}{x^2}} = \lim_{x\to 0} e^{\frac{\ln\cos x}{x^2}} = e^{\lim\limits_{x\to 0}\frac{\ln\cos x}{x^2}} \overset{(1)}{=} e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$$

$$\lim_{x\to 0}\frac{\ln\cos x}{x^2} \overset{\text{L'H}}{=} \lim_{x\to 0}\frac{-\sin x}{2x\cos x} = \lim_{x\to 0}\left(-\frac{1}{2\cos x}\right)\frac{\sin x}{x} = -\frac{1}{2} \tag{1}$$

7. Pomocou L'Hospitalovho pravidla:

$$\begin{aligned}
&\lim_{x\to 0}\frac{\sqrt{1+\operatorname{tg}x}-\sqrt{1+\sin x}}{x^3}\cdot\frac{\sqrt{1+\operatorname{tg}x}+\sqrt{1+\sin x}}{\sqrt{1+\operatorname{tg}x}+\sqrt{1+\sin x}}=\\
&\quad=\lim_{x\to 0}\frac{\operatorname{tg}x-\sin x}{x^3\left(\sqrt{1+\operatorname{tg}x}+\sqrt{1+\sin x}\right)}=\\
&\quad=\lim_{x\to 0}\frac{\operatorname{tg}x-\sin x}{x^3}\cdot\lim_{x\to 0}\frac{1}{\sqrt{1+\operatorname{tg}x}+\sqrt{1+\sin x}}\overset{(2,3)}{=}\frac{1}{4}
\end{aligned}$$

$$\lim_{x\to 0}\frac{1}{\sqrt{1+\operatorname{tg}x}+\sqrt{1+\sin x}}=\frac{1}{2} \tag{2}$$

$$\begin{aligned}
&\lim_{x\to 0}\frac{\operatorname{tg}x-\sin x}{x^3}\overset{\text{L'H}}{=}\lim_{x\to 0}\frac{\frac{1}{\cos^2 x}-\cos x}{3x^2}=\lim_{x\to 0}\frac{1-\cos^3 x}{3x^2\cos^2 x}\overset{\text{L'H}}{=}\\
&\quad=\lim_{x\to 0}\frac{3\cos^2 x\sin x}{6x\cos^2 x-6x^2\cos x\sin x}=\lim_{x\to 0}\frac{\cos^2 x\sin x}{2x\cos^2 x-x^2\sin 2x}\overset{\text{L'H}}{=}\\
&\quad=\lim_{x\to 0}\frac{-2\cos x\sin^2 x+\cos^3 x}{2\cos^2 x-4x\cos x\sin x-2x\sin 2x-2x^2\cos 2x}=\frac{1}{2}
\end{aligned} \tag{3}$$

Pomocou Taylorovych polynómov:

$$\begin{aligned}
\sqrt{1+\operatorname{tg}x}&=1+\frac{x}{2}-\frac{x^2}{8}+\frac{11x^3}{48}+o(x^3)\\
\sqrt{1+\sin x}&=1+\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{48}+o(x^3)
\end{aligned}$$

$$\begin{aligned}
&\lim_{x\to 0}\frac{\sqrt{1+\operatorname{tg}x}-\sqrt{1+\sin x}}{x^3}=\\
&=\lim_{x\to 0}\frac{\left(1+\frac{x}{2}-\frac{x^2}{8}+\frac{11x^3}{48}+o(x^3)\right)-\left(1+\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{48}+o(x^3)\right)}{x^3}=\lim_{x\to 0}\frac{\frac{x^3}{4}+o(x^3)}{x^3}=\\
&=\lim_{x\to 0}\left(\frac{1}{4}+\frac{o(x^3)}{x^3}\right)=\frac{1}{4}
\end{aligned}$$