Houston Journal of Mathematics

Volume 37, No. 1, 2011

FINITE-VALUED MAPPINGS PRESERVING DIMENSION

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Communicated by Yasunao Hattori

Abstract. We say that a set-valued mapping $F : X \Rightarrow Y$ is $\mathcal{C}$-*lsc* provided that there exists a countable cover $\mathcal{C}$ of X consisting of functionally closed sets such that for every $C \in \mathcal{C}$ and each functionally open set $U \subseteq Y$ one can find a functionally open set $V \subseteq X$ such that $\{x \in C : F(x) \cap U \neq \emptyset\} = C \cap V$. For Tychonoff spaces X and Y we write $X \rhd Y$ provided that there exist a finite-valued $\mathcal{C}$-lsc mapping $F : X \Rightarrow Y$ and a finite-valued $\mathcal{D}$-lsc mapping $G : Y \Rightarrow X$ (for suitable $\mathcal{C}$ and $\mathcal{D}$) such that $y \in \bigcup\{F(x) : x \in G(y)\}$ for every $y \in Y$. We prove that $X \rhd Y$ implies $\dim X \geq \dim Y$. (Here $\dim X$ denotes the Čech-Lebesgue (covering) dimension of X.) As a corollary, we obtain that $\dim X = \dim Y$ whenever a perfectly normal space Y is an image of a Tychonoff space X under a finite-to-one open mapping. We also give an example of an open mapping $f : X \to Y$ such that $|f^{-1}(y)| \leq 2$ for all $y \in Y$, both X and Y are hereditarily normal (and Y is even Lindelöf) but $\dim X \neq \dim Y$.

1. Introduction

Notation and terminology follows [4] if not stated otherwise. By dimension is meant the Čech-Lebesgue (covering) dimension dim. The set of all natural numbers is denoted by $\mathbb{N}$.

Let X, Y be topological spaces and 2^Y a set of all nonempty subsets of Y. We call a mapping $F : X \to 2^Y$ a *set-valued mapping from X to Y* and denote this fact by the symbol $F : X \Rightarrow Y$. A mapping $F : X \Rightarrow Y$ is called *finite-valued*

2000 *Mathematics Subject Classification.* 54C10, 54C60, 54F45.
Key words and phrases. Lower semi-continuous mapping, Covering dimension, Open mapping, Perfectly normal space.

provided that $F(x)$ is finite for every $x \in X$. If for every open $U \subseteq Y$ the set

$$F^{-1}(U) = \{x \in X : F(x) \cap U \neq \emptyset\}$$

is open in X, then F is said to be *lower semi-continuous* (abbreviated by *lsc*).

Recall that a subset A of a space X is called *functionally open* (*functionally closed*) provided that there exists a continuous function $f : X \to [0,1]$ such that $A = f^{-1}((0,1])$ ($A = f^{-1}(0)$ respectively).

We call a countable union of (functionally) closed subsets of a space X a (*functionally*) F_σ subset of X. Recall that a bijection $f : X \to Y$ between Tychonoff spaces X and Y is a *first level Borel (Baire) isomorphism* provided that $f(B)$ is a (functionally) F_σ subset of Y for every (functionally) F_σ subset B of X and $f^{-1}(C)$ is a (functionally) F_σ subset of X for every (functionally) F_σ subset of Y.

Clearly, if X and Y are homeomorphic, then $\dim X = \dim Y$. In [7, 2, 13, 12] it was shown that, for special classes of spaces X and Y, the same conclusion $\dim X = \dim Y$ holds if one considers a bijection $f : X \to Y$ satisfying much weaker continuity assumptions than that of a homeomorphism. The most general result in the series [7, 2, 13] says that if both X and Y are countable unions of (pseudo)compact spaces and there exists a first level Baire isomorphism $f : X \to Y$, then $\dim X = \dim Y$ [13]. Moreover, Pytkeev [12] proved that if X and Y are normal spaces and $f : X \to Y$ is a first level Borel isomorphism, then f is also a first level Baire isomorphism. It now follows that if $f : X \to Y$ is a first level Borel isomorphism between σ-compact Tychonoff spaces X and Y, then $\dim X = \dim Y$ [12]. In general, first level Borel isomorphisms do not preserve dimension [12].

In this paper we obtain another generalization of the notion of a homeomorphism by introducing a special class of *set-valued* mappings $F : X \Rightarrow Y$ such that one can *always* get the conclusion $\dim X = \dim Y$ for *arbitrary* Tychonoff spaces X and Y. To motivate this generalization, observe that spaces X and Y are homeomorphic if and only if there exist continuous mappings $f : X \to Y$ and $g : Y \to X$ such that

(1) $$x = g \circ f(x) \text{ for all } x \in X, \text{ and } y = f \circ g(y) \text{ for all } y \in Y.$$

(Indeed, (1) is equivalent to $g = f^{-1}$.) One obtains a straightforward generalization of a homeomorphism by replacing f and g with lower semi-continuous set-valued mappings $F : X \Rightarrow Y$ and $G : Y \Rightarrow X$ and requiring them to satisfy

(2) $$x \in G \circ F(x) \text{ for all } x \in X, \text{ and } y \in F \circ G(y) \text{ for all } y \in Y,$$

where the composition $G \circ F : X \Rightarrow X$ is defined by

$$G \circ F(x) = \bigcup \{G(y) : y \in F(x)\} \text{ for every } x \in X.$$

Note that a mapping $f : X \to Y$ is continuous if and only if the set-valued mapping $f' : X \Rightarrow Y$ defined by $f'(x) = \{f(x)\}$ for $x \in X$, is lower semi-continuous. This justifies the lower semi-continuity requirement on F and G above.

One cannot expect much from such a far-reaching generalization of a homeomorphism unless some additional conditions on images $F(x)$ and $G(y)$ of points $x \in X$ and $y \in Y$ are imposed. Indeed, take a singleton $\{x\}$ as X, the closed unit interval $[0, 1]$ as Y, and define $F(x) = Y$ and $G(y) = \{x\}$ for every $y \in Y$. Then both $F : X \Rightarrow Y$ and $G : Y \Rightarrow X$ are *compactly-valued* lower semi-continuous mappings satisfying the condition (2), and yet $\dim X = 0 \neq 1 = \dim Y$. Since compactness of the images of points does not suffice for preservation of dimension, we will further strengthen our condition by requiring images of points to be *finite*. This leads us to the question that motivated this manuscript: If lower semi-continuous finite-valued mappings $F : X \Rightarrow Y$ and $G : Y \Rightarrow X$ satisfy the condition (2), is then $\dim X = \dim Y$?

Unfortunately, as we will see in Example 6.3, the answer to this question is negative even in the special case when spaces X and Y are hereditarily normal. To rectify this situation one needs a strengthening of lower semi-continuity due to Gutev:

Definition 1. (Gutev [6]) A set-valued mapping $F : X \Rightarrow Y$ is called *strongly lower semi-continuous* (abbreviated *strongly lsc*) provided that the set $F^{-1}(U)$ is functionally open in X for every functionally open subset U of Y.

Note that, for a Tychonoff space Y, a mapping $f : X \to Y$ is continuous if and only if the set-valued mapping $f' : X \Rightarrow Y$ defined by $f'(x) = \{f(x)\}$ for $x \in X$, is strongly lower semi-continuous, so we can modify our question as follows:

Question 1. Suppose that X and Y are Tychonoff spaces. If strongly lower semi-continuous finite-valued mappings $F : X \Rightarrow Y$ and $G : Y \Rightarrow X$ satisfy (2), must $\dim X = \dim Y$?

We will provide a positive answer to this question in Corollary 5.2. Moreover, it is possible to get the equation $\dim X = \dim Y$ even under substantially weaker assumption obtained by considering a countable cover $\mathcal{C}$ of X consisting of functionally closed (and sort of C^*-embedded) subsets of X, a countable cover $\mathcal{D}$ of Y consisting of functionally closed (and sort of C^*-embedded) subsets of Y, and finite-valued mappings $F : X \Rightarrow Y$ and $G : Y \Rightarrow X$ such that the restriction $F\restriction_C$ of F to every $C \in \mathcal{C}$ and the restriction $G\restriction_D$ of G to every $D \in \mathcal{D}$ are strongly lower semi-continuous (Corollary 4.9).

Since the covering dimension $\dim X$ of a Tychonoff space X is defined by means of *functionally* open covers of X, it is not at all surprising that functionally open sets play such a prominent role in the following definition:

Definition 2. Suppose that X and Y are topological spaces, and $\mathcal{C}$ is a countable cover of X consisting of functionally closed subsets of X. We will say that a set-valued mapping $F : X \Rightarrow Y$ is *$\mathcal{C}$-lower semi-continuous* (abbreviated by *$\mathcal{C}$-lsc*) provided that for every $C \in \mathcal{C}$ and every functionally open set $U \subseteq Y$ there exists a functionally open set $V \subseteq X$ such that $F^{-1}(U) \cap C = V \cap C$.

From now on, when speaking about $\mathcal{C}$-lsc mapping $F : X \Rightarrow Y$, we will always assume that $\mathcal{C}$ is some countable functionally closed cover of X.

Clearly, if a set-valued mapping $F : X \Rightarrow Y$ is $\mathcal{C}$-lsc, then the restriction of F to each $C \in \mathcal{C}$ is strongly lower semi-continuous. It is also clear that a set-valued mapping $F : X \Rightarrow Y$ is $\{X\}$-lsc if and only if it is strongly lower semi-continuous.

Definition 3. Given topological spaces X and Y and finite-valued mappings $F : X \Rightarrow Y$ and $G : Y \Rightarrow X$, we write $X \,{}_F{\rhd}_G\, Y$ provided that F is $\mathcal{C}$-lsc for some $\mathcal{C}$, G is $\mathcal{D}$-lsc for some $\mathcal{D}$ and for every $y \in Y$ there exists $x \in X$ with $x \in G(y)$ and $y \in F(x)$.

Observe that the condition "for every $y \in Y$ there exists $x \in X$ with $x \in G(y)$ and $y \in F(x)$" in the above definition is equivalent to the condition "$y \in F \circ G(y)$ for every $y \in Y$". Therefore, if $X \,{}_F{\rhd}_G\, Y$ and $Y \,{}_G{\rhd}_F\, X$ hold at the same time, then F and G satisfy the condition (2).

Definition 4. We say that X *dominates* Y, and write $X \rhd Y$, provided that $X \,{}_F{\rhd}_G\, Y$ for some finite-valued mappings $F : X \Rightarrow Y$ and $G : Y \Rightarrow X$.

Our principal result in this paper is Theorem 4.8: If X and Y are Tychonoff spaces such that $X \rhd Y$, then $\dim X \geq \dim Y$. In particular, if X and Y are Tychonoff spaces, $X \rhd Y$ and $Y \rhd X$, then $\dim X = \dim Y$ (Corollary 4.9). The particular case of Theorem 4.8 when X and Y are separable metrizable is proved in Section 2 (Lemma 2.6). Applying some standard factorization machinery recalled in Section 3 we then prove the general case in Section 4. In Section 5 we highlight the connections between our results and strongly lsc mappings in the sense of Gutev [6] and σ-lsc mappings in the sense of Choban [3]. Finally, in Section 6 we apply our results to establish theorems about preservation of dimension under finite-to-one open mappings. In particular, we prove the following:

(i) If $f : X \to Y$ is a finite-to-one functionally open (=cozero) mapping[1] between Tychonoff spaces X and Y, then $\dim X = \dim Y$ (Corollary 6.2).

(ii) If a perfectly normal space Y is an image of a Tychonoff space X under a finite-to-one open mapping, then $\dim X = \dim Y$ (Corollary 6.3).

(iii) There exist hereditarily normal spaces X and Y and a finite-to-one open mapping $f : X \to Y$ such that $\dim X \neq \dim Y$ (Example 6.2).

2. Separable metrizable case

In this section we will prove that if X, Y are separable metrizable spaces and $X \,{}_F\!\rhd_G\, Y$, then $\dim X \geq \dim Y$.

Lemma 2.1. *Let X and Y be topological spaces such that Y is Tychonoff and $F : X \Rightarrow Y$ a $\mathcal{C}$-lsc mapping. Then, for every $C \in \mathcal{C}$, the restriction $F\restriction_C : C \to Y$ of F to C is an lsc mapping.*

Proof. Let $C \in \mathcal{C}$. Let W be an open subset of Y. Define

$$\mathcal{U} = \{U : U \text{ is a functionally open subset of } Y \text{ such that } U \subseteq W\}.$$

Since F is a $\mathcal{C}$-lsc mapping, for every $U \in \mathcal{U}$ there exists a functionally open subset V_U of X such that $(F\restriction_C)^{-1}(U) = F^{-1}(U) \cap C = V_U \cap C$. In particular, $(F\restriction_C)^{-1}(U)$ is open in C. Since $W = \bigcup \mathcal{U}$, we conclude that $(F\restriction_C)^{-1}(W) = \bigcup\{(F\restriction_C)^{-1}(U) : U \in \mathcal{U}\}$ is an open subset of C. Thus $F\restriction_C$ is lsc. □

Recall that if open subsets of a space X are F_σ-sets, then X is called a *perfect* space.

Lemma 2.2. *Let X be a perfect space and Y a Hausdorff space. Assume that $F : X \Rightarrow Y$ is an lsc mapping. Then $\{x \in X : |F(x)| = n\}$ is an F_σ-set for every $n \in \mathbb{N}$.*

Proof. First let us show that $U_n = \{x \in X : |F(x)| > n\}$ is open for every $n \in \mathbb{N}$. If $U_n = \emptyset$, we are done. Suppose now that $U_n \neq \emptyset$. Pick $x \in U_n$ arbitrarily. Since $|F(x)| > n$, there exist pairwise disjoint open subsets $V_1, \dots, V_{n+1}$ of Y such that $F(x) \cap V_i \neq \emptyset$ for every $i \leq n+1$. Since F is lsc, $W = \bigcap\{F^{-1}(V_i) : i = 1, \dots, n+1\}$ is an open neighborhood of x. We claim that $W \subseteq U_n$. Indeed, assume that $z \in W$. Then $F(z) \cap V_i \neq \emptyset$ for every $i \leq n+1$, which yields $|F(z)| > n$. Therefore $z \in U_n$. We have proved that U_n is open in X.

[1]Recall that f is *functionally open* iff f is continuous and $f(U)$ is a functionally open subset of Y for every functionally open $U \subseteq X$.

Being an open subset of a perfect space X, U_{n-1} is an F_σ-subset of X. Therefore, $\{x \in X : |F(x)| = n\} = (X \setminus U_n) \cap U_{n-1}$ is also an F_σ-subset of X. □

Lemma 2.3. *Let X and Y be topological spaces and $F : X \Rightarrow Y$ a $\mathcal{C}$-lsc mapping. If X is perfect and Y is Hausdorff, then $A_n = \{x \in X : |F(x)| = n\}$ is an F_σ-subset of X for every $n \in \mathbb{N}$.*

Proof. Fix $n \in \mathbb{N}$. Let $C \in \mathcal{C}$ be arbitrary. The restriction $F\restriction_C : C \Rightarrow Y$ of F to C is an lsc mapping (Lemma 2.1). Being a subspace of a perfect space X, C is also perfect. Applying Lemma 2.2 (to C taken as X and $F\restriction_C$ taken as F), we conclude that $C_n = \{x \in C : |F\restriction_C(x)| = n\} = \{x \in C : |F(x)| = n\}$ is an F_σ-subset of C. Since C is closed in X, C_n is an F_σ-subset of X as well.

Since $\mathcal{C}$ covers X, we have $A_n = \bigcup\{C_n : C \in \mathcal{C}\}$. Since $\mathcal{C}$ is countable, we conclude that A_n is an F_σ-subset of X. □

Lemma 2.4. *Let X be a Tychonoff and Y an arbitrary topological space, $\mathcal{B}$ a fixed base of X and $G : Y \Rightarrow X$ a $\mathcal{D}$-lsc mapping. Let $B_k = \{y \in Y : |G(y)| = k\}$. Suppose also that $D \in \mathcal{D}$, $y \in D \cap B_k$, $x \in G(y)$ and $V \subseteq X$ is an open neighborhood of x. Then there exist an open neighborhood $U \subseteq Y$ of y and $V' \in \mathcal{B}$ such that $x \in V' \subseteq V$ and $|G(y') \cap V'| = 1$ for every $y' \in D \cap B_k \cap U$.*

Proof. Let $\{x_1, \dots, x_k\} = G(y)$. Since $x \in G(y)$, there exists a unique $j \leq k$ such that $x = x_j$. Choose pairwise disjoint open sets $V_1, \dots, V_k \in \mathcal{B}$ such that $x_i \in V_i$ for $i = 1, \dots, k$. Without loss of generality we may assume that $V_j \subseteq V$. Define $U' = \bigcap\{(G\restriction_D)^{-1}(V_i) : i = 1, \dots, k\}$ and $V' = V_j$. Since $y \in D$ and $x_i \in G(y) \cap V_i$ for all $i = 1, \dots, k$, we have $y \in U'$. Since $G\restriction_D$ is lsc by Lemma 2.1, U' is an open subset of D. Choose an open subset U of Y with $U' = D \cap U$.

Suppose now that $y' \in D \cap B_k \cap U = B_k \cap U'$. If $i \leq k$, then $y' \in U' \subseteq (G\restriction_D)^{-1}(V_i)$ implies $G(y') \cap V_i \neq \emptyset$, and so we can choose $x'_i \in G(y') \cap V_i$. This gives $\{x'_1, \dots, x'_k\} \subseteq G(y')$. Since $V_1, \dots, V_k$ are pairwise disjoint, it follows that $x'_i \neq x'_l$ whenever $i, l \leq k$ and $i \neq l$. Since $y' \in B_k$, we conclude that $\{x'_1, \dots, x'_k\} = G(y')$. Thus $G(y') \cap V' = G(y') \cap V_j = \{x'_1, \dots, x'_k\} \cap V_j = \{x'_j\}$. □

Lemma 2.5. *Let X and Y be separable metrizable spaces such that $X\ {}_F\!\!\rhd_G Y$. Then there exists a countable cover $\mathcal{K}$ of Y such that each $K \in \mathcal{K}$ is closed and homeomorphic to some subspace of X.*

Proof. Let F and G be the mappings witnessing $X\ {}_F\!\!\rhd_G Y$, see Definition 3. Then F is $\mathcal{C}$-lsc and G is $\mathcal{D}$-lsc for some $\mathcal{C}, \mathcal{D}$. For $k, l \in \mathbb{N}$ define $A_l = \{x \in X : |F(x)| = l\}$ and $B_k = \{y \in Y : |G(y)| = k\}$. Let $\mathcal{C} = \{C_j : j \in \mathbb{N}\}$ and $\mathcal{D} = \{D_i : i \in \mathbb{N}\}$ be enumerations of $\mathcal{C}$ and $\mathcal{D}$, respectively. Fix a countable base $\{V_n : n \in \mathbb{N}\}$ of X and a countable base $\{U_m : m \in \mathbb{N}\}$ of Y.

Define $\mathcal{E}$ to be the set of all $\epsilon = (i, j, k, l, m, n, p) \in \mathbb{N}^7$ such that $V_p \subseteq V_n$, $|G(y) \cap V_n| = 1$ for every $y \in D_i \cap B_k \cap U_m$ and $|F(x) \cap U_m| = 1$ for every $x \in C_j \cap A_l \cap V_p$.

Fix $\epsilon = (i, j, k, l, m, n, p) \in \mathcal{E}$. Define mappings $f_\epsilon : C_j \cap A_l \cap V_p \to U_m$ and $g_\epsilon : D_i \cap B_k \cap U_m \to V_n$ by $f_\epsilon(x) = y$, where $\{y\} = F(x) \cap U_m$ and $g_\epsilon(y) = x$, where $\{x\} = G(y) \cap V_n$. Since $F\restriction_{C_j}$ and $G\restriction_{D_i}$ are lsc by Lemma 2.1, from our definition of f_ϵ and g_ϵ we conclude that both f_ϵ and g_ϵ are continuous.

We claim that

$$L_\epsilon = g_\epsilon^{-1}(C_j \cap A_l \cap V_p) \subseteq D_i \cap B_k \cap U_m \tag{3}$$

is an F_σ-subset of Y. Being a space with a countable base, X is perfect, and so A_l is an F_σ-subset of X by Lemma 2.3. Since C_j is closed in X and V_p is open in X, we conclude that $C_j \cap A_l \cap V_p$ is an F_σ-subset of X. Since g_ϵ is continuous, $g_\epsilon^{-1}(C_j \cap A_l \cap V_p)$ is an F_σ-subset of $D_i \cap B_k \cap U_m$. Applying Lemma 2.3 once again, we obtain that B_k is an F_σ-subset of Y. Since D_i is closed in Y and U_m is open in Y (and thus an F_σ-subset of Y), it follows that $D_i \cap B_k \cap U_m$ is an F_σ-subset of Y. As an (relative with the subspace topology) F_σ-subset of an F_σ-subset of Y, L_ϵ is also an F_σ-subset of Y.

Because the (continuous) composition $f_\epsilon \circ g_\epsilon$ is defined everywhere on L_ϵ, the set

$$K_\epsilon = \{y \in L_\epsilon : y = f_\epsilon \circ g_\epsilon(y)\} \tag{4}$$

is closed in L_ϵ. Hence, K_ϵ is an F_σ-subset of Y, and so $K_\epsilon = \bigcup\{K_\epsilon^s : s \in \mathbb{N}\}$, where each K_ϵ^s is closed in Y.

The family $\mathcal{K} = \{K_\epsilon^s : \epsilon \in \mathcal{E}, s \in \mathbb{N}\}$ is countable and consists of closed subsets of Y. For $\epsilon \in \mathcal{E}$ and $s \in \mathbb{N}$, from (3) and (4) it follows that g_ϵ is a homeomorphism between K_ϵ and $g_\epsilon(K_\epsilon)$, and so K_ϵ^s is homeomorphic to $g_\epsilon(K_\epsilon^s)$. It remains only to prove that $\mathcal{K}$ covers Y. To this end, it suffices to check that $Y = \bigcup_{\epsilon \in \mathcal{E}} K_\epsilon$. Fix $y \in Y$. According to our assumptions there exists $x \in X$ such that $y \in F(x)$ and $x \in G(y)$. Then $y \in D_i \cap B_k$ and $x \in C_j \cap A_l$ for some $i, j, k, l \in \mathbb{N}$. According to Lemma 2.4 there exists an open neighborhood $U \subseteq Y$ of y and $n \in \mathbb{N}$ such that $x \in V_n$ and $|G(y') \cap V_n| = 1$ for every $y' \in D_i \cap B_k \cap U$. Using Lemma 2.4 once again we can find $m \in \mathbb{N}$ and open $V \subseteq X$ such that $y \in U_m \subseteq U$, $x \in V$ and $|F(x') \cap U_m| = 1$ for every $x' \in C_j \cap A_l \cap V$. Then $x \in V_p \subseteq V_n \cap V$ for some $p \in \mathbb{N}$. Clearly then $|F(x') \cap U_m| = 1$ for every $x' \in C_j \cap A_l \cap V_p$. Now $\epsilon = (i, j, k, l, m, n, p) \in \mathcal{E}$. Finally, note that $g_\epsilon(y) = x$ and $f_\epsilon(x) = y$, which gives $y \in K_\epsilon$. □

Lemma 2.6. *Let X, Y be separable metrizable spaces such that $X\ {}_F{\rhd}_G\ Y$. Then $\dim X \geq \dim Y$.*

Proof. Apply Lemma 2.5 to find a countable cover $\mathcal{K}$ of Y such that every $K \in \mathcal{K}$ is closed in Y and homeomorphic to some subspace X_K of X. By the Subspace Theorem for dimension of metrizable spaces (see [4, Theorem 7.3.4]), $\dim K = \dim X_K \leq \dim X$. Now the Countable Sum Theorem (see [4, Theorem 7.2.1]) implies that $\dim Y \leq \dim X$. □

3. Standard factorization machinery

In this section we recall some standard factorization machinery. For a topological space X let $\mathbb{S}_X$ be the set of all continuous mappings from X into the Hilbert cube $I^{\aleph_0}$. For $f, g \in \mathbb{S}_X$ we write $f \preceq g$ if there exists a continuous mapping $h : g(X) \to f(X)$ such that $f = h \circ g$. One can easily check that the relation $\preceq$ on $\mathbb{S}_X$ is reflexive and transitive. However, $\preceq$ is not anti-symmetric. To fix this, we will introduce an appropriate quotient of $\mathbb{S}_X$. If $f, g \in \mathbb{S}_X$, $f \preceq g$ and $g \preceq f$, then we write $f \approx g$. Clearly, $\approx$ is an equivalence relation on $\mathbb{S}_X$. One can easily see that $f \approx g$ if and only iff there exists a homeomorphism $h : g(X) \to f(X)$ such that $f = h \circ g$.

As usual, $[g]_\approx = \{f \in \mathbb{S}_X : f \approx g\}$ denotes the equivalence class of $g \in \mathbb{S}_X$ with respect to the relation $\approx$. Define $\mathbb{P}_X = \{[g]_\approx : g \in \mathbb{S}_X\}$. We write $[f]_\approx \preceq [g]_\approx$ if $f \preceq g$. Clearly the relation "$\preceq$" on $\mathbb{P}_X$ is well defined and makes $\mathbb{P}_X$ into a partially ordered set (poset). With a certain abuse of notation, from now on we will not distinguish between $f \in \mathbb{S}_X$ and $[f]_\approx \in \mathbb{P}_X$. In particular, we will write $f \in \mathbb{P}_X$ instead of cumbersome $[f]_\approx \in \mathbb{P}_X$.

Suppose that $f : X \to f(X)$ is an arbitrary continuous mapping such that $f(X)$ is separable metrizable. Since Hilbert cube is universal for all separable metrizable spaces, there exists a homeomorphic embedding $h : f(X) \to I^{\aleph_0}$. Now $g = h \circ f \in \mathbb{S}_X$ (and with abuse of notation we have agreed upon also $g \in \mathbb{P}_X$). So f can be identified with its "representative" g in $\mathbb{P}_X$.

Lemma 3.1. *Assume that X is a topological space and $\mathcal{F} \subseteq \mathbb{P}_X$ is countable. Then:*

(i) $\triangle\mathcal{F} \in \mathbb{P}_X$,
(ii) *$f \preceq \triangle\mathcal{F}$ for every $f \in \mathcal{F}$,*
(iii) *if $g \in \mathbb{P}_X$ and $f \preceq g$ for every $f \in \mathcal{F}$, then $\triangle\mathcal{F} \preceq g$.*

Proof. (i) is clear. As $f = \pi_f \circ \triangle\mathcal{F}$, where $\pi_f : \prod\{f(X) : f \in \mathcal{F}\} \to f(X)$ is the projection, we get (ii). To prove (iii), let $g \in \mathbb{P}_X$ be such that $f \preceq g$

for every $f \in \mathcal{F}$. Then for every $f \in \mathcal{F}$ there exists a continuous mapping $h_f : g(X) \to f(X)$ such that $h_f \circ g = f$. The diagonal mapping $h = \triangle\{h_f : f \in \mathcal{F}\} : g(X) \to \prod\{f(X) : f \in \mathcal{F}\}$ is continuous and satisfies $h \circ g = \triangle\mathcal{F}$. This proves $\triangle\mathcal{F} \preceq g$. □

The following standard definition will find its use in simplifying some proofs in the sequel.

Definition 5. Let X be a topological space and $\mathcal{F} \subseteq \mathbb{P}_X$. We say that $\mathcal{F}$ is:

(i) *closed* (in $\mathbb{P}_X$) if for every sequence $\{f_n : n \in \mathbb{N}\} \subseteq \mathcal{F}$ such that $f_0 \preceq f_1 \preceq \dots \preceq f_n \preceq f_{n+1} \preceq \dots$ one has $\triangle\{f_n : n \in \mathbb{N}\} \in \mathcal{F}$,
(ii) *unbounded* (in $\mathbb{P}_X$) if for every $f \in \mathbb{P}_X$ there exists $g \in \mathcal{F}$ such that $f \preceq g$,
(iii) a *club* (in $\mathbb{P}_X$) if $\mathcal{F}$ is both closed and unbounded (in $\mathbb{P}_X$).

Our next lemma is a part of folklore. We include its proof for the reader's convenience.

Lemma 3.2. *Let X be a topological space and $\{\mathcal{F}_n : n \in \mathbb{N}\}$ a sequence of clubs in $\mathbb{P}_X$. Then $\mathcal{F} = \bigcap\{\mathcal{F}_n : n \in \mathbb{N}\}$ is a club in $\mathbb{P}_X$.*

PROOF. Obviously, $\mathcal{F}$ is closed. To prove that it is unbounded fix an arbitrary $f \in \mathbb{P}_X$. Since each $\mathcal{F}_n$ is unbounded, using the standard diagonal argument we can find a sequence $\{f_i : i \in \mathbb{N}\} \subseteq \mathbb{P}_X$ such that $f \preceq f_0 \preceq f_1 \preceq \dots \preceq f_n \preceq f_{n+1} \preceq \dots$ and the set $M_n = \{m \in \mathbb{N} : f_m \in \mathcal{F}_n\}$ is infinite for every $n \in \mathbb{N}$. Then $g = \triangle\{f_i : i \in \mathbb{N}\} \in \mathbb{P}_X$ by Lemma 3.1(i). From Lemma 3.1(ii) it follows that $f \preceq f_0 \preceq g$. So it remains only to show that $g \in \mathcal{F}$.

Fix $n \in \mathbb{N}$. Applying Lemma 3.1(ii) to f_m, we get $f_m \preceq g$ for all $m \in \mathbb{N}$. Thus $\triangle\{f_m : m \in M_n\} \preceq g$ by item (iii) of Lemma 3.1. Let $i \in \mathbb{N}$. Since M_n is infinite, there exists $m_0 \in M_n$ such that $f_i \preceq f_{m_0}$. Furthermore, $f_{m_0} \preceq \triangle\{f_m : m \in M_n\}$ by Lemma 3.1(ii). Applying Lemma 3.1(iii), we obtain $g \preceq \triangle\{f_m : m \in M_n\}$. Since $\triangle\{f_m : m \in M_n\} \preceq g \preceq \triangle\{f_m : m \in M_n\}$, we get $g = \triangle\{f_m : m \in M_n\}$. Since $\{f_m : m \in M_n\} \subseteq \mathcal{F}_n$ and $\mathcal{F}_n$ is closed, one has $\triangle\{f_m : m \in M_n\} \in \mathcal{F}_n$, which yields $g \in \mathcal{F}_n$.

Since $n \in \mathbb{N}$ was arbitrary, we conclude that $g \in \bigcap\{\mathcal{F}_n : n \in \mathbb{N}\} = \mathcal{F}$. □

We will need the following well-known lemma.

Lemma 3.3. *Let X be a Tychonoff space, $n \in \mathbb{N}$ and $S_n(X) = \{f \in \mathbb{P}_X : \dim f(X) \leq n\}$. Then*

(i) *$\dim X \leq n$ if and only if $S_n(X)$ is unbounded in $\mathbb{P}_X$,*
(ii) *if $\dim X \leq n$, then $S_n(X)$ is a club in $\mathbb{P}_X$.*

Proof. (i) is proved in [9, Theorem 2]. From (i) it follows that $S_n(X)$ is unbounded. The proof that $S_n(X)$ is closed can be found, for example, in [13, Lemma 11]. $\square$

Remark. If X and Y are topological spaces and $f \in \mathbb{P}_{X\oplus Y}$, we define $i(f) = (f{\restriction}_X, f{\restriction}_Y) \in \mathbb{P}_X \times \mathbb{P}_Y$. Observe that i is an order-preserving bijection between partially ordered sets $(\mathbb{P}_{X\oplus Y}, \preceq)$ and $(\mathbb{P}_X, \preceq) \times (\mathbb{P}_Y, \preceq)$. In view of this identification the notion of a club in $\mathbb{P}_X \times \mathbb{P}_Y$ is already defined.

4. General case

In this section we will extend Lemma 2.6 to arbitrary Tychonoff spaces X and Y, thereby proving our main result in this manuscript (see Theorem 4.8 and Corollary 4.9).

Lemma 4.1. *Suppose that $\mathcal{C}$ is a countable cover of a topological space X consisting of functionally closed subsets of X. Assume also that Y is a separable metric space, $\mathcal{O}$ is a countable base of Y and $F : X \Rightarrow Y$ is a set-valued mapping satisfying the following condition: For every $C \in \mathcal{C}$ and every $O \in \mathcal{O}$ there exists a functionally open set $V_O \subseteq X$ such that $F^{-1}(O) \cap C = V_O \cap C$. Then F is $\mathcal{C}$-lsc.*

Proof. Let U be a functionally open subset of Y. Since $\mathcal{O}$ is a base for Y, there exists $\mathcal{O}_U \subseteq \mathcal{O}$ with $U = \bigcup \mathcal{O}_U$. As a countable union of functionally open subsets of X, the set $V = \bigcup\{V_O : O \in \mathcal{O}_U\}$ is functionally open in X. Finally, note that

$$F^{-1}(U) \cap C = \bigcup\{F^{-1}(O) : O \in \mathcal{O}_U\} \cap C = \bigcup\{(F^{-1}(O) \cap C : O \in \mathcal{O}_U\} =$$

$$\bigcup\{V_O \cap C : O \in \mathcal{O}_U\} = \bigcup\{V_O : O \in \mathcal{O}_U\} \cap C = V \cap C$$

$\square$

Definition 6. For a mapping $f : X \to Z$ and a family $\mathcal{C} \subseteq 2^X$ we will use the symbol $f(\mathcal{C})$ to denote the family $\{f(C) : C \in \mathcal{C}\} \subseteq 2^Z$.

Our next lemma, which plays the key role in our factorization, was inspired by [6, Theorem 1.1].

Lemma 4.2. *Let X be a topological space, Y a separable metrizable space and $F : X \Rightarrow Y$ a set-valued mapping such that $F(x)$ is closed in Y for every $x \in X$. Then the following are equivalent:*

(i) *F is $\mathcal{C}$-lsc.*

(ii) *There exist a separable metrizable space Z, a continuous mapping $h : X \to Z$ and an $h(\mathcal{C})$-lsc mapping $\varphi : Z \Rightarrow Y$ such that $F = \varphi \circ h$ and $C = h^{-1}(h(C))$ for each $C \in \mathcal{C}$.*

Proof. (i)⇒(ii) Fix a countable base $\{U_i : i \in \mathbb{N}\}$ of the topology of Y and let $\mathcal{C} = \{C_n : n \in \mathbb{N}\}$. Since F is $\mathcal{C}$-lsc, for every $i, n \in \mathbb{N}$ there exists a functionally open set $V_{in} \subseteq X$ such that $V_{in} \cap C_n = F^{-1}(U_i) \cap C_n$. Let $f_{in} : X \to [0,1]$ be a continuous function such that $V_{in} = f_{in}^{-1}((0,1])$. Further, since each C_n is functionally closed, there exists a continuous function $g_n : X \to [0,1]$ such that $C_n = g_n^{-1}(0)$. Define $\mathcal{F} = \{f_{in} : i, n \in \mathbb{N}\} \bigcup \{g_n : n \in \mathbb{N}\}$. The diagonal mapping $h = \triangle\mathcal{F}$ is continuous and $Z = h(X)$ is a separable metric space.

Claim 1. *If $x_0, x_1 \in X$ and $h(x_0) = h(x_1)$, then $F(x_0) = F(x_1)$.*

Proof. Indeed, $h(x_0) = h(x_1)$ implies

$$f(x_0) = f(x_1) \text{ for every } f \in \mathcal{F}. \tag{5}$$

Since $\{C_n : n \in \mathbb{N}\}$ covers X, there exists $n \in \mathbb{N}$ such that $x_0 \in C_n$. From $g_n \in \mathcal{F}$ and (5) we get $g_n(x_1) = g_n(x_0) = 0$, and so $x_1 \in C_n$ as well.

Assume, by the way of contradiction, that $F(x_0) \neq F(x_1)$. We may assume, without loss of generality, that there exists $y \in F(x_1) \setminus F(x_0)$. Since $F(x_0)$ is closed in Y, there exists $i \in \mathbb{N}$ such that $y \in U_i$ and $U_i \cap F(x_0) = \emptyset$.

Since $y \in F(x_1) \cap U_i$ and $x_1 \in C_n$, we get $x_1 \in F^{-1}(U_i) \cap C_n = V_{in} \cap C_n \subseteq V_{in} = f_{in}^{-1}((0,1])$, and so $f_{in}(x_1) \neq 0$. Since $U_i \cap F(x_0) = \emptyset$, one has $x_0 \notin F^{-1}(U_i)$. Since $x_0 \in C_n$ and $F^{-1}(U_i) \cap C_n = V_{in} \cap C_n$, it follows that $x_0 \notin V_{in} = f_{in}^{-1}((0,1])$, and therefore $f_{in}(x_0) = 0$. We have proved that $f_{in}(x_0) \neq f_{in}(x_1)$. This contradicts $f_{in} \in \mathcal{F}$ and (5). □

Let $z \in Z$. Choose arbitrarily $x \in h^{-1}(z)$ and put $\varphi(z) = F(x)$. By Claim 1, $\varphi(z)$ does not depend on the choice of $x \in h^{-1}(z)$. This defines a set-valued mapping $\varphi : Z \Rightarrow Y$ satisfying $F = \varphi \circ h$.

Fix $n \in \mathbb{N}$. Let $\theta_n : Z \to g_n(X)$ be the continuous mapping such that $g_n = \theta_n \circ h$. Since $C_n = g_n^{-1}(0) = (\theta_n \circ h)^{-1}(0) = h^{-1}(\theta_n^{-1}(0))$, the set $h(C_n) = \theta_n^{-1}(0)$ is closed in Z and $C_n = h^{-1}(h(C_n))$. Since Z is a separable metric space, $h(C_n)$ is functionally closed. Furthermore, $Z = h(X) = h(\bigcup\{C_n : n \in \mathbb{N}\}) = \bigcup\{h(C_n) : n \in \mathbb{N}\}$. We have proved that $h(\mathcal{C})$ is a countable functionally closed cover of Z and $C = h^{-1}(h(C))$ for each $C \in \mathcal{C}$. Let $i, n \in \mathbb{N}$. Let $\psi_{in} : Z \to f_{in}(X)$ be the continuous mapping such that $f_{in} = \psi_{in} \circ h$. Then $W_{in} = \psi_{in}^{-1}((0,1])$ is an open subset of Z. Note that

$$V_{in} = f_{in}^{-1}((0,1]) = (\psi_{in} \circ h)^{-1}((0,1]) = h^{-1}(\psi_{in}^{-1}((0,1]) = h^{-1}(W_{in}),$$

and so $h(V_{in}) = W_{in}$. Let $Z_n = h(C_n)$. Since $F = \varphi \circ h$, we have

$$h^{-1}(\varphi^{-1}(U_i) \cap Z_n) = h^{-1}(\varphi^{-1}(U_i)) \cap h^{-1}(Z_n) = (\varphi \circ h)^{-1}(U_i) \cap C_n =$$

$$F^{-1}(U_i) \cap C_n = V_{in} \cap C_n = V_{in} \cap h^{-1}(Z_n).$$

Therefore, $\varphi^{-1}(U_i) \cap Z_n = h(V_{in} \cap h^{-1}(Z_n)) = h(V_{in}) \cap Z_n = W_{in} \cap Z_n$. Being an open subset of the separable metric space Z, W_{in} is functionally open in Z.

We have checked that φ satisfies the assumptions of Lemma 4.1. Applying this lemma, we conclude that φ is $h(\mathcal{C})$-lsc.

(ii)⇒(i) Since $h(\mathcal{C})$ is a countable (functionally) closed cover of Z, $C = h^{-1}(h(C))$ for each $C \in \mathcal{C}$ and h is continuous, $\mathcal{C}$ is a countable functionally closed cover of X.

Fix $C \in \mathcal{C}$ and a (functionally) open set $U \subseteq Y$. Since φ is $h(\mathcal{C})$-lsc, there exists a (functionally) open set $V \subseteq Z$ such that $\varphi^{-1}(U) \cap h(C) = V \cap h(C)$, and consequently

$$F^{-1}(U) \cap C = (\varphi \circ h)^{-1} \cap C = h^{-1}(\varphi^{-1}(U)) \cap h^{-1}(h(C)) = h^{-1}(\varphi^{-1}(U) \cap h(C)) =$$

$$h^{-1}(V \cap h(C)) = h^{-1}(V) \cap h^{-1}(h(C)) = h^{-1}(V) \cap C.$$

Since h is continuous, $h^{-1}(V)$ is a functionally open subset of X. Hence F is $\mathcal{C}$-lsc. □

Lemma 4.3. *Let X be a topological space, Y and Z be separable metrizable spaces, $f : X \to Y$ a continuous mapping and $F : X \rightrightarrows Z$ a $\mathcal{C}$-lsc finite-valued mapping. Then the diagonal product $G = f \triangle F : X \rightrightarrows Y \times Z$ which assigns to every $x \in X$ the set $G(x) = \{f(x)\} \times F(x)$, is a finite-valued $\mathcal{C}$-lsc mapping.*

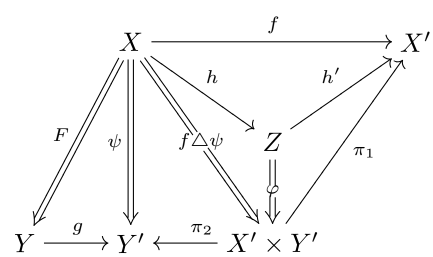
Proof. Let $\mathcal{B}_Y$ and $\mathcal{B}_Z$ be countable bases of Y and Z, respectively. Then $\mathcal{B} = \{V \times W : V \in \mathcal{B}_Y, W \in \mathcal{B}_Z\}$ is a countable base of the product $Y \times Z$. Fix $C \in \mathcal{C}$, $V \in \mathcal{B}_Y$ and $W \in \mathcal{B}_Z$. Since f is continuous, $f^{-1}(V)$ is functionally open in X. Since F is $\mathcal{C}$-lsc, there exists a functionally open subset U' of X such that $C \cap F^{-1}(W) = C \cap U'$. Then $U = U' \cap f^{-1}(V)$ is a functionally open subset of X such that $C \cap G^{-1}(V \times W) = C \cap F^{-1}(W) \cap f^{-1}(V) = C \cap U' \cap f^{-1}(V) = C \cap U$. It remains only to apply Lemma 4.1. □

Lemma 4.4. *Suppose that X is a topological space, Y and Z are separable metrizable spaces. Assume that $f : X \to Y$ and $g : X \to Z$ are continuous surjections, and $h : Z \to Y$ is a (not necessarily continuous) mapping satisfying $f = h \circ g$. Then there exists a stronger separable metrizable topology τ on Z with respect to which h becomes and g remains continuous.*

Proof. Denote by $\mathcal{B}_Y$ and $\mathcal{B}_Z$ countable bases for Y and Z, respectively. Then $\mathcal{S} = \mathcal{B}_Z \cup \{h^{-1}(U) : U \in \mathcal{B}_Y\}$ is a subbase for some new topology τ on Z. This topology can be described also as a subspace topology of $i\triangle h(Z) \subseteq Z \times Y$, where $i : Z \to Z$ is the identity mapping and Z as a set is identified with $i\triangle h(Z)$. Therefore τ is separable metrizable. It is clear that $h : (Z, \tau) \to Y$ is continuous. To observe that $g : X \to (Z, \tau)$ is continuous take $V \in \mathcal{S}$. If $V \in \mathcal{B}_Z$, then $g^{-1}(V)$ is open because g was continuous in the original topology. If $V = h^{-1}(U)$ for some $U \in \mathcal{B}_Y$, then $g^{-1}(V) = f^{-1}(U)$ is open due to the continuity of f. □

Lemma 4.5. *Let X, Y be topological spaces, $F : X \Rightarrow Y$ a finite-valued $\mathcal{C}$-lsc mapping, $f \in \mathbb{P}_X$ and $g \in \mathbb{P}_Y$. Then there exist $h \in \mathbb{P}_X$ and an $h(\mathcal{C})$-lsc mapping $F' : h(X) \Rightarrow g(Y)$ such that $f \preceq h$, $g \circ F = F' \circ h$ and $h^{-1}(h(C)) = C$ for every $C \in \mathcal{C}$.*

Proof. Define $X' = f(X)$ and $Y' = g(Y)$. Denote by $\pi_1 : X' \times Y' \to X'$, $\pi_2 : X' \times Y' \to Y'$ the projections. It is clear that $\psi = g \circ F : X \Rightarrow Y'$ is a $\mathcal{C}$-lsc mapping. It follows from Lemma 4.3 that the mapping $f\triangle\psi : X \Rightarrow X' \times Y'$ is $\mathcal{C}$-lsc as well. Use Lemma 4.2 to find a separable metrizable space Z, a continuous mapping $h : X \to Z$ and an $h(\mathcal{C})$-lsc mapping $\varphi : Z \Rightarrow X' \times Y'$ such that $f\triangle\psi = \varphi \circ h$ and $h^{-1}(h(C)) = C$ for every $C \in \mathcal{C}$. Let $h' = \pi_1 \circ \varphi$. Since $f = \pi_1 \circ (f\triangle\psi) = \pi_1 \circ (\varphi \circ h) = (\pi_1 \circ \varphi) \circ h = h' \circ h$ is a (single-valued) mapping, the mapping h' must be single-valued, and thus h' may be viewed as a mapping from Z to X'. The reader may want to consult the following commutative diagram:



According to Lemma 4.4, there exists some stronger separable metrizable topology τ on Z such that h remains and h' becomes continuous with respect to τ. Consider Z with τ. Then $f \preceq h$. Clearly φ remains $h(\mathcal{C})$-lsc in the stronger topology τ. Let $F' = \pi_2 \circ \varphi$. Clearly, F' is an $h(\mathcal{C})$-lsc mapping satisfying $g \circ F = F' \circ h$. □

Lemma 4.6. *Let X, Y be topological spaces and $F : X \Rightarrow Y$ a finite-valued $\mathcal{C}$-lsc mapping. Then the set S_F of all pairs $(f, g) \in \mathbb{P}_X \times \mathbb{P}_Y$ satisfying conditions (i) and (ii) below forms a club in $\mathbb{P}_X \times \mathbb{P}_Y$.*

(i) $C = f^{-1}(f(C))$ *for every* $C \in \mathcal{C}$,

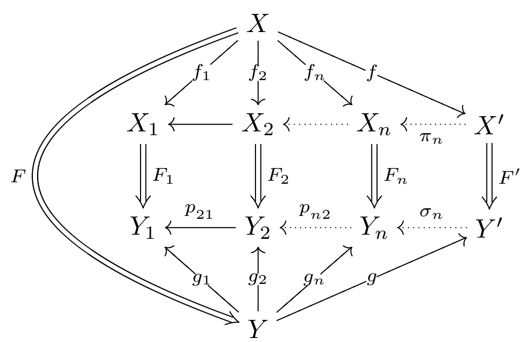
(ii) *there exists a finite-valued* $f(\mathcal{C})$*-lsc mapping* $F' : f(X) \Rightarrow g(Y)$ *such that* $g \circ F = F' \circ f$.

Proof. For $(f, g) \in \mathbb{P}_X \times \mathbb{P}_Y$, let $h \in \mathbb{P}_X$ be as in the conclusion of Lemma 4.5. Then $(f, g) \preceq (h, g)$ and $(h, g) \in S_F$, which proves that S_F is unbounded.

To show that S_F is closed, choose an increasing chain $\{(f_i, g_i) \in \mathbb{P}_X \times \mathbb{P}_Y : i \in \mathbb{N}\} \subseteq S_F$. That is, $(f_i, g_i) \preceq (f_j, g_j)$ whenever $i \leq j$, and for every $X_i = f_i(X)$ there is an $f_i(\mathcal{C})$-lsc mapping $F_i : X_i \Rightarrow Y_i = g_i(Y)$ such that $g_i \circ F = F_i \circ f_i$. Define $f = \triangle\{f_i : i \in \mathbb{N}\}$, $g = \triangle\{g_i : i \in \mathbb{N}\}$, $X' = f(X)$ and $Y' = g(Y)$. We have to prove that $(f, g) = \triangle\{(f_i, g_i) : i \in \mathbb{N}\} \in S_F$. To this end, we need to show that f and g satisfy items (i) and (ii) of our lemma.

(i) Observe that $f_1 \preceq f$ by Lemma 3.1(ii), so there exists a (continuous) mapping $h : f(X) \to f_1(X)$ such that $f_1 = h \circ f$. Then $f^{-1}(D) = f_1^{-1}(h(D))$ for every $D \subseteq f(X)$. Applying this formula to $D = f(C)$ for $C \in \mathcal{C}$, we obtain $f^{-1}(f(C)) = f_1^{-1}(h(f(C))) = f_1^{-1}(f_1(C)) = C$ because f_1 satisfies (i). This shows that f satisfies (i) as well.

(ii) We are going to find a finite-valued $f(\mathcal{C})$-lsc mapping $F' : X' \Rightarrow Y'$ making the following diagram commutative:



Let $\alpha_n : \prod\{X_i : i \in \mathbb{N}\} \to X_n$ and $\beta_n : \prod\{Y_i : i \in \mathbb{N}\} \to Y_n$ be the projections. For every $n \in \mathbb{N}$, define $\pi_n = \alpha_n \restriction_{X'}$ and $\sigma_n = \beta_n \restriction_{Y'}$.

Claim 2. *Assume that* $x_0, x_1 \in X$ *and* $f(x_0) = f(x_1)$. *Then* $g \circ F(x_0) = g \circ F(x_1)$.

Proof. If $y_0, y_1 \in Y'$ and $y_0 \neq y_1$, then there exists $n(y_0, y_1) \in \mathbb{N}$ such that $\sigma_n(y_0) \neq \sigma_n(y_1)$ for all $n \geq n(y_0, y_1)$. Since F is finite-valued, the set $T = g \circ F(x_0) \cup g \circ F(x_1)$ is finite, and so $n = \max\{n(y_0, y_1) : y_0, y_1 \in T, y_0 \neq y_1\} \in \mathbb{N}$. Clearly, $\sigma_n \restriction_T : T \to Y_n$ is an injection. Now $f(x_0) = f(x_1)$ implies $f_n(x_0) =$

$f_n(x_1)$, which gives $g_n \circ F(x_0) = F_n \circ f_n(x_0) = F_n \circ f_n(x_1) = g_n \circ F(x_1)$. Since $g_n = \sigma_n \circ g$, we get $\sigma_n(g \circ F(x_0)) = \sigma_n(g \circ F(x_1))$, and so $g \circ F(x_0) = g \circ F(x_1)$ because $\sigma_n \restriction_T$ is an injection. $\square$

For $x' \in X'$ pick some $x \in X$ such that $f(x) = x'$ and define $F'(x') = g(F(x))$. Claim 2 guarantees that the value $F'(x')$ does not depend on the choice of $x \in f^{-1}(x')$. Hence F' is a well defined finite-valued mapping that makes the above diagram commutative. In particular, $g \circ F = F' \circ f$. It remains only to show that F' is $f(\mathcal{C})$-lsc.

Since $(f_1, g_1) \in S_F$, $f_1(\mathcal{C})$ is a countable cover consisting of functionally closed subsets of $X_1 = f_1(X)$. Since the mapping h from the proof of item (i) is continuous, $h^{-1}(C) = f(C)$ is a functionally closed subset of $X' = f(X)$ for every $C \in \mathcal{C}$. Therefore $f(\mathcal{C})$ is a countable (functionally) closed cover of X'.

For $i, j \in \mathbb{N}$ with $j < i$ one has $g_j \preceq g_i$, so we can fix a continuous function $p_{ij} : Y_i \to Y_j$ such that $p_{ij} \circ g_i = g_j$. For every $i \in \mathbb{N}$ let $\mathcal{B}_i$ be a countable base of Y_i. Without loss of generality, we may assume that $\{p_{ij}^{-1}(B) : B \in \mathcal{B}_j\} \subseteq \mathcal{B}_i$ whenever $j < i$. Then $\mathcal{B} = \bigcup\{\{\sigma_n^{-1}(B) : B \in \mathcal{B}_n\} : n \in \mathbb{N}\}$ is a countable base of Y'.

Let $C \in \mathcal{C}$ and $U \in \mathcal{B}$. Then there exist $n \in \mathbb{N}$ and $U_n \in \mathcal{B}_n$ such that $U = \sigma_n^{-1}(U_n)$. Since F_n is $f_n(\mathcal{C})$-lsc, there exists a functionally open subset V_n of X_n such that $f_n(C) \cap F_n^{-1}(U_n) = f_n(C) \cap V_n$. Since π_n is continuous, $V = \pi_n^{-1}(V_n)$ is a functionally open subset of X'. Now we obtain the following chain of equalities:

$$F'^{-1}(U) \cap f(C) \overset{(i)}{=} \pi_n^{-1}(F_n^{-1}(U_n)) \cap f(C) \overset{(ii)}{=} \pi_n^{-1}(F_n^{-1}(U_n) \cap f_n(C)) = \pi_n^{-1}(V_n \cap f_n(C)) = V \cap f(C).$$

Indeed, the equality (i) follows from $\sigma_n \circ F' = F_n \circ \pi_n$ and $U = \sigma_n^{-1}(U_n)$, and the equality (ii) follows from $\pi_n^{-1}(f_n(C)) = f(C)$.

Applying Lemma 4.1, we conclude that F' is $f(\mathcal{C})$-lsc. $\square$

Lemma 4.7. *Let $n \in \mathbb{N}$. Suppose that X and Y are Tychonoff spaces such that $\dim X \leq n$ and $X\ {}_F\!\rhd_G\ Y$ (see Definition 3). Let $S_n(X)$ be the subset of $\mathbb{P}_X$ defined in Lemma 3.3. Let $S_F \subseteq \mathbb{P}_X \times \mathbb{P}_Y$ and $S_G \subseteq \mathbb{P}_Y \times \mathbb{P}_X$ be the sets defined in Lemma 4.6. Let $j : \mathbb{P}_Y \times \mathbb{P}_X \to \mathbb{P}_X \times \mathbb{P}_Y$ be the isomorphism defined by $j(g, f) = (f, g)$ for $(g, f) \in \mathbb{P}_Y \times \mathbb{P}_X$. Finally, define*

$$A(n, F, G) = (S_n(X) \times \mathbb{P}_Y) \cap S_F \cap j(S_G).$$

Then:

(i) *$A(n, F, G)$ is a club in $\mathbb{P}_X \times \mathbb{P}_Y$, and*

(ii) $(f,g) \in A(n,F,G)$ *implies* $\dim g(Y) \le n$.

Proof. (i) Since $S_n(X)$ is a club in $\mathbb{P}_X$ by Lemma 3.3(ii), $S_n(X) \times \mathbb{P}_Y$ is a club in $\mathbb{P}_X \times \mathbb{P}_Y$. According to lemma 4.6, S_F is a club in $\mathbb{P}_X \times \mathbb{P}_Y$ and S_G is a club in $\mathbb{P}_Y \times \mathbb{P}_X$. Since j is an isomorphism, $j(S_G)$ is a club in $\mathbb{P}_X \times \mathbb{P}_Y$. Now the conclusion of item (i) follows from Lemma 3.2 (applied to $\mathbb{P}_{X\oplus Y}$, see remark following Lemma 3.3).

(ii) Let $F : X \Rightarrow Y$ be $\mathcal{C}$-lsc for some $\mathcal{C}$ and $G : Y \Rightarrow X$ be $\mathcal{D}$-lsc for some $\mathcal{D}$. Since $(f,g) \in S_F$, there exists a finite-valued $f(\mathcal{C})$-lsc mapping $F' : f(X) \Rightarrow g(Y)$ such that $g \circ F = F' \circ f$. Since $(f,g) \in j(S_G)$, we have $(g,f) \in S_G$, and so there exists a finite-valued $g(\mathcal{D})$-lsc mapping $G' : g(Y) \Rightarrow f(X)$ such that $f \circ G = G' \circ g$.

Let $y' \in g(Y)$ be arbitrary. Pick $y \in Y$ such that $g(y) = y'$. Since $X \,{}_F{\rhd}_G\, Y$, there exists $x \in X$ with $x \in G(y)$ and $y \in F(x)$. Define $x' = f(x) \in f(X)$. Now $x' = f(x) \in f(G(y)) = f \circ G(y) = G' \circ g(y) = G'(g(y)) = G'(y')$ and $y' = g(y) \in g(F(x)) = g \circ F(x) = F' \circ f(x) = F'(f(x)) = F'(x')$.

We have proved that $f(X) \,{}_{F'}{\rhd}_{G'}\, g(Y)$. Therefore, $\dim g(Y) \le \dim f(X)$ by Lemma 2.6. Finally, $(f,g) \in (S_n(X) \times \mathbb{P}_Y)$ implies $f \in S_n(X)$, and so $\dim f(X) \le n$. This proves $\dim g(Y) \le n$. □

Theorem 4.8. *Let X and Y be Tychonoff spaces such that $X \rhd Y$. Then* $\dim X \ge \dim Y$.

Proof. There exist finite-valued mappings $F : X \Rightarrow Y$ and $G : Y \Rightarrow X$ such that $X \,{}_F{\rhd}_G\, Y$. The case $\dim X = \infty$ is trivial. Suppose now that $\dim X = n$ for some $n \in \mathbb{N}$. Let $g_0 \in \mathbb{P}_Y$. Pick an arbitrary $f_0 \in \mathbb{P}_X$. Since the set $A(n,F,G)$ is unbounded in $\mathbb{P}_X \times \mathbb{P}_Y$ by Lemma 4.7(i), we can find some pair $(f,g) \in A(n,F,G)$ such that $(f_0, g_0) \preceq (f,g)$. Then $g_0 \preceq g$ and $\dim g(Y) \le n$ by Lemma 4.7(ii). This argument shows that $S_n(Y)$ is unbounded in $\mathbb{P}_Y$. Therefore, $\dim Y \le n$ by Lemma 3.3(i). □

As a direct corollary we obtain that the notion of mutual domination is the desired generalization of homeomorphism of spaces which preserves dimension.

Corollary 4.9. *If X and Y are Tychonoff spaces such that $X \rhd Y$ and $Y \rhd X$, then* $\dim X = \dim Y$.

Some other corollaries and consequences of Theorem 4.8 will be discussed in the next two sections.

5. Applications to strongly lsc mappings in the sense of Gutev and σ-lsc mappings in the sense of Choban

Corollary 5.1. *Suppose that X and Y are Tychonoff spaces, $F : X \Rightarrow Y$ and $G : Y \Rightarrow X$ are finite-valued mappings such that for every $y \in Y$ there exists $x \in X$ with $x \in G(y)$ and $y \in F(x)$. If both F and G are strongly lower semi-continuous, then* $\dim Y \leq \dim X$.

Proof. Since F is strongly lsc, F is $\{X\}$-lsc. Similarly, since G is strongly lsc, G is $\{Y\}$-lsc. Now the conclusion follows from Theorem 4.8. □

From the last corollary we immediately get

Corollary 5.2. *Suppose that X_0 and X_1 are Tychonoff spaces, and $F_i : X_i \Rightarrow X_{1-i}$ is a finite-valued strongly lsc mapping for every $i = 0, 1$. Suppose also that $x \in \bigcup\{F_{1-i}(y) : y \in F_i(x)\}$ whenever $i = 0, 1$ and $x \in X_i$. Then* $\dim X_0 = \dim X_1$.

Recall that a subspace C of a topological space X is called *$C^\star$-embedded* in X if every bounded continuous real-valued function on C can be continuously extended over X.

Corollary 5.3. *Suppose that X and Y are normal spaces, $F : X \Rightarrow Y$ and $G : Y \Rightarrow X$ are finite-valued mappings such that for every $y \in Y$ there exists $x \in X$ with $x \in G(y)$ and $y \in F(x)$. Assume also that $\mathcal{C}$ is a countable cover of X consisting of functionally closed subsets of X such that the restriction $F \restriction_C : C \Rightarrow Y$ of F to each $C \in \mathcal{C}$ is strongly lower semi-continuous. Similarly, suppose that $\mathcal{D}$ is a countable cover of Y consisting of functionally closed subsets of Y such that the restriction $G \restriction_D : D \Rightarrow X$ of G to each $D \in \mathcal{D}$ is strongly lower semi-continuous. Then* $\dim Y \leq \dim X$.

Proof. Let $C \in \mathcal{C}$ be arbitrary. As a closed subset of a normal space X, C is C^*-embedded in X. In particular, if W is a functionally open subset of C, then one can find a functionally open subset V of X with $V \cap C = W$. This shows that F is $\mathcal{C}$-lsc. A similar argument shows that G is $\mathcal{D}$-lsc. Hence $X\ {}_F{\rhd}_G\ Y$, and the conclusion of our corollary follows from Theorem 4.8. □

In [3] Choban introduced the notion of a σ-lower semi-continuous mapping. A set-valued mapping $F : X_0 \Rightarrow X_1$ between Tychonoff spaces X_0 and X_1 is called *σ-lower semi-continuous* provided that there exists some countable closed cover $\mathcal{C}$ of X_0 such that the restriction $F \restriction_C : C \Rightarrow Y$ of F to each $C \in \mathcal{C}$ is lower semi-continuous [3].

Corollary 5.4. *Suppose that X and Y are perfectly normal spaces, $F : X \Rightarrow Y$ and $G : Y \Rightarrow X$ are finite-valued σ-lower semi-continuous mappings such that for every $y \in Y$ there exists $x \in X$ with $x \in G(y)$ and $y \in F(x)$. Then* $\dim Y \leq \dim X$.

Proof. Let $\mathcal{C}$ be a cover witnessing that F is σ-lower semi-continuous, and let $\mathcal{D}$ be a cover witnessing that G is σ-lower semi-continuous. Since both X and Y are perfectly normal, each member of $\mathcal{C}$ is functionally closed in X and each member of $\mathcal{D}$ is functionally closed in Y. Every open subset of a perfectly normal space is functionally open, and so a lower semi-continuous set-valued mapping defined on a perfectly normal space is strongly lower semi-continuous. Therefore, all the assumptions of Corollary 5.3 are satisfied. Thus $\dim Y \leq \dim X$ by Corollary 5.3. □

According to Choban [3], spaces X_0 and X_1 are called *σ-om-equivalent* (*om-equivalent*) provided that, for every $i \in \{0, 1\}$, there exists a finite-valued σ-lsc mapping (lsc mapping) $F_i : X_i \Rightarrow X_{1-i}$ such that $x \in \bigcup\{F_{1-i}(y) : y \in F_i(x)\}$ for every $x \in X_i$. Using this terminology, from Corollary 5.4 we immediately get the following

Corollary 5.5. *Let X and Y be σ-om-equivalent perfectly normal spaces. Then* $\dim X = \dim Y$.

Example 6.3(ii) below shows that "perfectly normal" cannot be weakened to "hereditarily normal" in Corollary 5.5.

Corollary 5.6. (Choban, see [3, Corollary 4.4.6]) *If X and Y are σ-om-equivalent hereditary Lindelöf spaces, then* $\dim X = \dim Y$. [2]

6. Applications to finite-to-one open mappings

A classical result of Pears [8] says that if f is a finite-to-one open mapping of a weakly paracompact normal space X onto a normal space Y, then $\dim X = \dim Y$. In this section we will use Corollary 4.9 to derive similar kind of theorems under different assumptions on X and Y. We will also provide two examples showing that, in general, dimension is not preserved by finite-to-one open mappings.

We start with a rather general theorem.

[2] This result in [3] relies on a proposition [3, Proposition 4.1.5] having an extremely condensed proof which the author had difficulties to follow.

Theorem 6.1. *Let X, Y be Tychonoff spaces and $f : X \to Y$ a finite-to-one mapping which is onto. Suppose that there exist countable functionally closed covers $\mathcal{C}$ of X and $\mathcal{D}$ of Y such that the following conditions are satisfied:*

(i) *for every $C \in \mathcal{C}$ and every functionally open $V \subseteq Y$ there exists functionally open $U \subseteq X$ such that $f^{-1}(V) \cap C = U \cap C$,*
(ii) *for every $D \in \mathcal{D}$ and every functionally open $U \subseteq X$ there exists a functionally open $V \subseteq Y$ such that $D \cap f(U) = D \cap V$.*

Then $\dim X = \dim Y$.

Proof. Define $F : X \Rightarrow Y$ and $G : Y \Rightarrow X$ by $F(x) = \{f(x)\}$ for $x \in X$ and $G(y) = f^{-1}(y)$ for $y \in Y$. A straightforward check shows that $X\ {}_F{\rhd}_G\ Y$ and $Y\ {}_G{\rhd}_F\ X$. Hence $X \rhd Y$ and $Y \rhd X$, and so $\dim X = \dim Y$ by Corollary 4.9. □

Recall that a mapping $f : X \to Y$ is *functionally open* (also called a *cozero mapping* in [1]) provided that f is continuous and the image $f(U)$ of every functionally open subset U of X is a functionally open subset of Y.

Corollary 6.2. *Let $f : X \to Y$ be a finite-to-one functionally open mapping of a Tychonoff space X onto a Tychonoff space Y. Then* $\dim X = \dim Y$.

Proof. Apply Theorem 6.1 to $\mathcal{C} = \{X\}$ and $\mathcal{D} = \{Y\}$. □

Examples 6.1 and 6.2 below show that "functionally open" cannot be weakened to "open" in Corollary 6.2.

Corollary 6.3. *Let $f : X \to Y$ be a finite-to-one open mapping of a Tychonoff space X onto a perfectly normal space Y. Then* $\dim X = \dim Y$.

Proof. Observe that every open mapping onto a perfectly normal space is functionally open and apply Corollary 6.2. □

Example 6.2 below shows that "perfectly normal" cannot be weakened to "hereditarily normal" in Corollary 6.2.

Let $\{U_a : a \in A\}$ be a family of subsets of a topological space Y. For every $a \in A$ let $i_a : U_a \to Y$ be the inclusion mapping (defined by $i_a(x) = x$ for $x \in U_a$). The mapping $f : \bigoplus\{U_a : a \in A\} \to Y$ defined by $f(x) = i_a(x)$ for $x \in X$, where a is the unique element of A with $x \in U_a$, will be called the *natural mapping* of $\bigoplus\{U_a : a \in A\}$ into Y. The following easy lemma provides a source of examples of finite-to-one open mappings.

Lemma 6.4. *Let $\{U_a : a \in A\}$ be a point-finite open cover of a topological space Y. Then the natural mapping $f : \bigoplus\{U_a : a \in A\} \to Y$ is a finite-to-one open mapping which is onto.*

Let us mention one folklore consequence of Corollary 6.3:

Corollary 6.5. *Let* $\{U_a : a \in A\}$ *be a point-finite open cover of a perfectly normal space* Y *such that* $\dim U_a \le n$ *for every* $a \in A$. *Then* $\dim Y \le n$.

Proof. Define $X = \bigoplus\{U_a : a \in A\}$. Clearly $\dim X \le n$. The natural mapping $f : X \to Y$ is open, finite-to-one and onto (Lemma 6.4). Since Y is perfectly normal, it follows from Corollary 6.3 that $\dim Y \le n$. □

It should be noted that it is known that in Corollary 6.5 it suffices to require Y to be hereditarily normal. Indeed, by [5], Problem 3.1.A.(b) there exists an open cover $\{V_a : a \in A\}$ of Y such that $\overline{V_a} \subseteq U_a$ for every $a \in A$. By Theorem 3.1.3 in [5], $\dim \overline{V_a} \le n$ and by Theorem 3.1.13 of [5], $\dim Y \le n$.

Our next two examples show that dimension need not be preserved by finite-to-one open mappings. Both examples are based on constructions of E.Pol and R.Pol. The first example shows that finite-to-one open mappings can raise dimension, while the second one shows that they can also lower dimension.

Recall that a space X is *scattered* if every subset of X has an isolated point.

Example 6.1. (R.Pol) *There exists a finite-to-one open mapping* f *from a scattered paracompact space* X *onto a weakly paracompact Tychonoff space* Y *such that* $0 = \dim X < \dim Y$.

Proof. According to [10, Example 4.C] there exists a weakly paracompact Tychonoff space Y such that $\dim Y > 0$ and $Y = W \cup D$, where both W and D are discrete subsets of Y and W is open in Y. Therefore, for every $y \in Y$ there exists an open neighborhood V_y of y such that V_y has at most one non-isolated point. Since Y is weakly paracompact, we can choose a point-finite refinement $\{U_a : a \in A\}$ of the open cover $\{V_y : y \in Y\}$ of Y. It follows that each U_a has at most one non-isolated point. Being a disjoint sum of spaces with at most one non-isolated point, $X = \bigoplus\{U_a : a \in A\}$ is a scattered paracompact space with $\dim X = 0$. Finally, the natural mapping $f : X \to Y$ is finite-to-one, open and onto (Lemma 6.4). □

We note that the theorem of Pears [8] cited in the beginning of this section shows that the space Y in the above example cannot be normal.

Example 6.2. *There exist a Lindelöf hereditarily normal space* Y *with* $\dim Y = 0$ *and, for every* $n \in \mathbb{N}$, *an open mapping* $f_n : X_n \to Y$ *of a hereditarily normal space* X_n *onto* Y *such that* $\dim X_n = n$ *and* $|f_n^{-1}(y)| \le 2$ *for all* $y \in Y$.

Proof. According to [11, Theorem 3], there exists a Lindelöf hereditary normal space Y such that $\dim Y = 0$ and Y contains, for every $n \in \mathbb{N}$, a subspace Y_n with $\dim Y_n = n$. By checking the proof of this theorem one concludes that Y_n is moreover open in Y for every $n \in \mathbb{N}$. Put $X_n = Y \oplus Y_n$. Since Y is hereditarily normal, so is X_n. Clearly $\dim X_n = n$. By Lemma 6.4 the natural mapping $f_n : X_n \to Y$ is open and onto. Clearly, $|f_n^{-1}(y)| \leq 2$ for all $y \in Y$. □

Our last example shows that Corollary 5.5 does not hold for arbitrary Tychonoff spaces.

Example 6.3. *For $i = 1, 2$ there exist om-equivalent (and thus σ-om-equivalent) Tychonoff spaces X_i and Y_i such that $\dim X_i \neq \dim Y_i$ and:*

(i) *X_1 is scattered and paracompact, while Y_1 is weakly paracompact,*
(ii) *both X_2 and Y_2 are hereditarily normal, and Y_2 is Lindelöf.*

Proof. Let a Tychonoff space Y be an image of a Tychonoff space X under a finite-to-one open mapping $f : X \to Y$. Define $F : X \Rightarrow Y$ and $G : Y \Rightarrow X$ by $F(x) = \{f(x)\}$ for $x \in X$ and $G(y) = f^{-1}(y)$ for $y \in Y$. It is easy to check that F and G are witnessing the *om*-equivalence of X and Y. Now it suffices to let $X_1 = X$, $Y_1 = Y$, where X and Y are the spaces from Example 6.1, and $X_2 = X_{278}$, $Y_2 = Y$, where X_{278} and Y are the spaces from Example 6.2. □

Let Y be the space Y constructed in Example 6.2. We note that there exists a point $y \in Y$ such that $Y \setminus \{y\}$ is perfectly normal and locally second countable (thus metrizable). This means that, in a certain sense, Y is very close to being perfectly normal and yet Y is an image under a finite-to-one open mapping of a space having different dimension. Together with Corollary 6.3 this makes it natural to ask the following

Question 2. *Are the following statements equivalent for a Tychonoff space Y?*

(i) *Y is perfectly normal.*
(ii) *$\dim X = \dim Y$ whenever $f : X \to Y$ is a finite-to-one open mapping of a Tychonoff space X onto Y.*

Acknowledgement

The author would like to thank Professor Dmitri Shakhmatov for helpful discussions of the topic, as well as for providing a valuable advice resulted in a substantial improvement of the exposition.

The author would like to express his gratitude to Professor Roman Pol who kindly communicated Example 6.1 to the author during the 49th Workshop "Advances in Set-Theoretic Topology: Conference in Honor of Tsugunori Nogura on his 60th Birthday" of the International School of Mathematics "G.Stampacchia" held on June 9-19, 2008 at the Center for Scientific Culture "Ettore Majorana" in Erice, Sicily.

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Received August 25, 2008

Revised version received December 24, 2008

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